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AN EXTENSION OF JUNG'S THEOREM

BY

B. V. DEKSTER[†]

Department of Mathematics, Erindale College of University of Toronto, Mississauga, Ontario L5L 1C6, Canada

ABSTRACT

THEOREM. *Let a set $X \subset R^n$ have unit circumradius and let B be the unit ball containing X. Put $C = \operatorname{conv} \bar{X}$, $D = \operatorname{diam} C$ ($= \operatorname{diam} X$), $k = \dim C$, $d_i = \sqrt{(2i+2)/i}$. Then*: (i) $D \in [d_n, 2]$; (ii) $k \geqq m$ *where* $m \in \{2, 3, \ldots, n\}$ *satisfies* $D \in [d_m, d_{m-1})$ (d_i *decreases by* i); (iii) *In case* $k = m$ (*by* (ii), *this is always the case when* $m = n$), *C contains a k-simplex* Δ *such that*: (α) *its vertices are on* ∂B; (β) *the centre of B belongs to* $\operatorname{int} \Delta$; ($\gamma$) *the inequalities* $\lambda_k(D) \leqq l \leqq D$ *with*

$$\lambda_k(D) = D \sqrt{\frac{4k - 2D^2(k-1)}{2 - (k-2)(D^2 - 2)}}, \qquad D \in [d_k, d_{k-1}),$$

are unimprovable estimates for length l of any edge of Δ.

§1. Introduction

1.1. Jung's Theorem [1, Theorem 2.6] applied to compact convex sets can be stated as follows.

THEOREM 1.1. *Let $C \subset R^n$ be a compact convex set of unit circumradius and let B be the unit ball containing C. Then* $\operatorname{diam} C$ *is not less than* $\sqrt{(2n+2)/n}$ (*which is the edge length of a regular n-simplex s inscribed in B*). *If* $\operatorname{diam} C = \sqrt{(2n+2)/n}$ *then C contains such a simplex s.*

A few questions can be raised in this connection. Suppose $\operatorname{diam} C$ is greater than but close to $\sqrt{(2n+2)/n}$. One might feel that C should then contain an n-simplex s' inscribed in B and close to s. How close? What is the range of edge length of s' in terms of $\operatorname{diam} C$?

Since inradius r and minimal width W of C are not less than those of s', one has thus lower bounds for r and W in terms of $\operatorname{diam} C$. (After dilatation, these lower bounds will depend on circumradius, too.) What are these lower bounds?

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When diam C ($\leqq 2$) is large enough then dim C (dimension of C) can be less than n, so that the lower bounds above turn into zeros. But then clearly dim C can be estimated from below in terms of diam C. It is obvious, for example, that $\dim C > 1$ if $\operatorname{diam} C < 2$. Indeed, otherwise the segment $C \subset B$ should be a diameter of B (because C has unit circumradius) and thus diam $C = 2$. What is the lower bound for dim C?

Theorem 1.2 and Remark 2 in 1.4 will answer those questions. Some of the relations obtained are new even for $n = 2$ (see Remark 3 in 1.4).

1.2. We denote by $\bar{X}$ the closure of X. By conv, int, diam, dim and vert, we mean convex hull, interior, diameter, dimension and set of vertices, respectively.

THEOREM 1.2. *Let a set $X \subset R^n$ have unit circumradius and let B be the unit ball containing X. Put $C = \operatorname{conv} \bar{X}$, $D = \operatorname{diam} C$ $(= \operatorname{diam} X)$, $k = \dim C$, $d_i = \sqrt{(2i+2)/i}$. Then*

(i) $D \in [d_n, 2]$;

(ii) $k \geqq m$ *where* $m \in \{2, 3, \ldots, n\}$ *satisfies* $D \in [d_m, d_{m-1})$ (d_i *is a decreasing function of* i);

(iii) *in case* $k = m$ *(by* (ii), *this is always the case when* $m = n$), *C contains a k-simplex Δ such that*

(α) *its vertices are on* ∂B;

(β) *the center of B belongs to* $\operatorname{int} \Delta$;

(γ) *the inequalities* $\lambda_k(D) \leqq l \leqq D$ *with*

$$\lambda_k(D) = D \sqrt{\frac{4k - 2D^2(k-1)}{2 - (k-2)(D^2 - 2)}}, \qquad D \in [d_k, d_{k-1}) \tag{1}$$

are unimprovable estimates for length l of any edge of Δ.

REMARKS. (1) Note that among all k-simplexes $\Delta \subset C$ satisfying (α) and (β), there might be none of diameter D.

(2) According to Theorem 1.3 (C), (D), $\lambda_k(D)$ has the following interpretation. Consider a k-simplex $\sigma_k(D)$ inscribed in a unit k-dimensional ball B_k having all edges but one of length $D \in [d_k, d_{k-1})$. Then $\lambda_k(D)$ is the length of that exceptional edge. Obviously $\sigma_k(D)$ is the convex hull of two regular $(k-1)$-simplexes δ' and δ'' with edge length D and with their vertices on ∂B_k having a common $(k-2)$-face δ. The edge having length $\lambda_k(D)$ is that one opposing δ (see Fig. 1).

It is obvious now that $\lambda_n(d_n) = d_n$. As D grows within $[d_n, d_{n-1})$, the quantity $\lambda_n(D)$ clearly decreases and vanishes as $D \to d_{n-1}$. In that process, the n-simplex

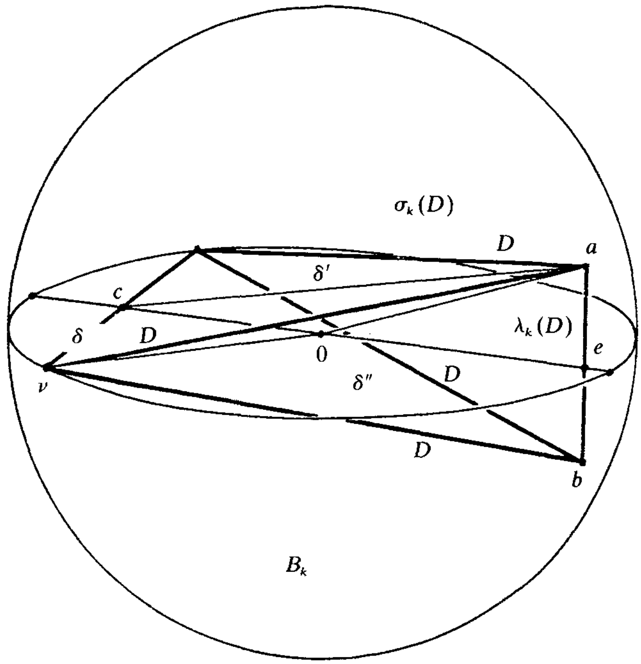

Fig. 1.

$\sigma_n(D)$ flattens and degenerates into a regular $(n-1)$-simplex inscribed in B_{n-1} when $D = d_{n-1}$. At that moment, λ_n is not defined anymore but $\lambda_{n-1}(d_{n-1}) = d_{n-1}$. Similarly, as D grows within $[d_{n-1}, d_{n-2})$, we deal with $\sigma_{n-1}(D)$. It flattens into a regular $(n-2)$-simplex while $\lambda_{n-1}(D)$ decreases and vanishes as $D \to d_{n-2}$; and so on. Eventually, when D grows within $[d_2, d_1) = [\sqrt{3}, 2)$, our $\sigma_2(D)$ is an isosceles triangle whose equal sides have length D. It degenerates into a diameter of the circle B_2 (and B_n) while $\lambda_2(D)$ decreases from $\sqrt{3}$ to 0. (See Fig. 2 for the graph of λ_k.)

(3) Equalities in both estimates (γ) clearly take place for $C = \sigma_k(D)$.

(4) Jung's Theorem is a combination of (i) with the particular case of (iii) when $D = d_n$ (because in that case, $m = k = n$ according to (ii), and $\lambda_n(d_n) = d_n$).

1.3. *Basic notation and some relations.* The segment with ends x and y, as well as the length of that segment, will be denoted by xy. For a compact closed convex set $C \subset R^k$, we denote by $W_k(C)$ and $r_k(C)$ the minimal width and inradius respectively of C in R^k. (Thus $r_k(C) = W_k(C) = 0$ if $\dim C < k$.)

As above, $d_k\ (=\sqrt{(2k+2)/k})$ denotes diameter of a regular k-simplex s_k

inscribed in a unit k-dimensional ball B_k. For $D \in (0, d_{k-1})$ and $k \geqq 2$, put

$$(2)\quad \Delta_k(D) = \left\{ \operatorname{conv} \bigcup_{i=1}^{k+1} v_i \;\middle|\; v_i \in \partial B_k,\ 0 \in \operatorname{conv} \bigcup_{i=1}^{k+1} v_i,\ \operatorname{diam} \operatorname{conv} \bigcup_{i=1}^{k+1} v_i \leqq D \right\},$$

where 0 is the center of B_k.

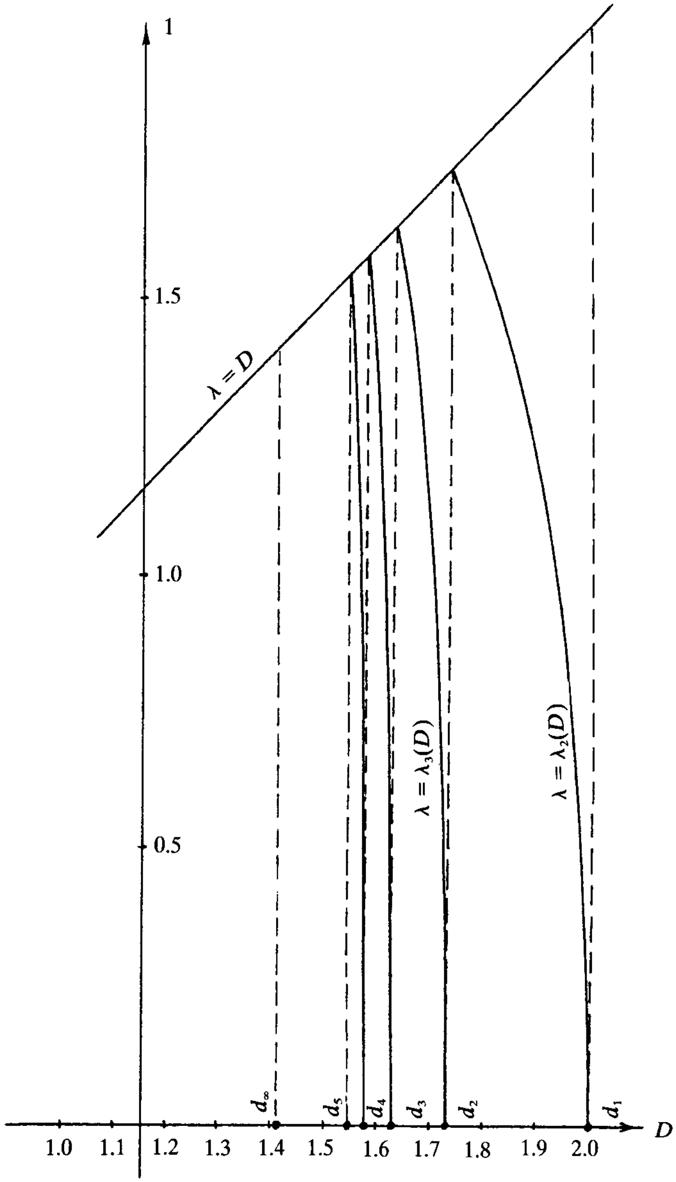

Fig. 2.

For an element $\Delta = \operatorname{conv} \bigcup_{i=1}^{k+1} v_i \in \Delta_k(D)$, put $l(\Delta) = \min_{1 \leqq i < j \leqq k+1} v_i v_j$. Now put

$$\lambda_k(D) = \min\{l(\Delta) \mid \Delta \in \Delta_k(D)\};$$

$$\rho_k(D) = \min\{r_k(\Delta) \mid \Delta \in \Delta_k(D)\};$$

$$\omega_k(D) = \min\{W_k(\Delta) \mid \Delta \in \Delta_k(D)\}.$$

(The minima exist by reason of compactness.)

Theorem 1.2 is deduced in §2 from the following

THEOREM 1.3. (A) *$\Delta_k(D)$ is empty for $D < d_k$ and non-empty for $D \in [d_k, d_{k-1})$. Each element of $\Delta_k(d_k)$ is isometric to s_k.*

(B) *Each $\Delta \in \Delta_k(D)$ is a k-simplex and $0 \in \operatorname{int} \Delta$ for $D \in [d_k, d_{k-1})$.*

(C) *If $l(\Delta) = \lambda_k(D)$ for some $\Delta \in \Delta_k(D)$ then all edges of Δ except one have length D while the exceptional one has length $\lambda_k(D)$. (Thus $\Delta = \sigma_k(D)$ as defined in Remark* (2) *in* 1.2.)

(D) *$\lambda_k(D)$ here is given by the formula* (1).

Theorem 1.3 is proved in §3. We prove also in §4 the following

LEMMA 1.3.

(3) $$\rho_2(D) = \frac{D^2\sqrt{1-D^2/4}}{2+2\sqrt{1-D^2/4}}, \qquad D \in [\sqrt{3}, 2);$$

(4) $$\omega_2(D) = D^2\sqrt{1-D^2/4}, \qquad D \in [\sqrt{3}, 2).$$

The lower bounds $\rho_2(D)$ and $\omega_2(D)$ are attained by an isosceles triangle $\sigma_2(D) \in \Delta_2(D)$ whose equal sides have length D.

(5) $$\rho_k(D) \geqq \sqrt{1-D^2(k-1)/2k}, \qquad k \geqq 2, \quad D \in [d_k, d_{k-1});$$

(6) $$\omega_k(D) > 2\sqrt{1-D^2(k-1)/2k}, \qquad k \geqq 2, \quad D \in [d_k, d_{k-1}).$$

1.4. REMARKS. (1) We do not know exact expressions for $\rho_k(D)$ and $\omega_k(D)$, when $k \geqq 3$.

(2) Let us return to statement (iii) of Theorem 1.2. Due to (α) and (β), the k-simplex Δ there belongs to $\Delta_k(D)$. Since

$$r_k(C) \geqq r_k(\Delta) \geqq \rho_k(D) \qquad \text{and} \qquad W_k(C) \geqq W_k(\Delta) \geqq \omega_k(D),$$

the relations (3)–(6) imply

$$r_k(C) \geqq \begin{cases} \dfrac{D^2\sqrt{1-D^2/4}}{2+2\sqrt{1-D^2/4}} & \text{if } k=2, \quad (7) \\ \sqrt{1-D^2(k-1)/2k} & \text{if } k \geqq 3; \quad (8) \end{cases}$$

$$W_k(C) \begin{cases} \geqq D^2\sqrt{1-D^2/4} & \text{if } k=2, \quad (9) \\ >2\sqrt{1-D^2(k-1)/2k} & \text{if } k \geqq 3. \quad (10) \end{cases}$$

By Lemma 1.3, equalities in (7) and (9) take place for $C=\sigma_2(D)$.

(3) Let $n=2$ in Theorem 1.2. If $D\neq 2$, one has $k=m=2$. (For $D=2$, m is not defined.) However, formulas (7) and (9) clearly work for any $D\in[\sqrt{3},2]$. After dilatation, they yield:

$$r \geqq \frac{d^2\sqrt{R^2-d^2/4}}{2R(R+\sqrt{R^2-d^2/4})}, \tag{11}$$

$$W \geqq \frac{d^2}{R^2}\sqrt{R^2-d^2/4}, \tag{12}$$

where W, d, r and R are minimal width, diameter, inradius and circumradius of an arbitrary convex figure in a plane different from a point. The equalities take place in (11) and (12) for any isosceles triangle whose equal sides are not shorter than the third side.

The inequality (11) was established by Santaló [3] in 1959. He also anticipated a relation of type (12) but could not derive it; see [3, pp. 102–103]. (Thus, the unknown curve ST in [3, Fig. 14] is given by $x=2y^2\sqrt{1-y^2}$.) More inequalities involving W, d, r and R can be found in [4, 5, 6].

§2. Proof of Theorem 1.2

2.1. We fisrt prove the statement 1.2(i). Let points $v_1, v_2, \dots, v_{n+1}\in C\cap\partial B$, not necessarily distinct, be such that the distance between $\delta=\operatorname{conv}\bigcup_{i=1}^{n+1} v_i$ and the center 0 of B is the least over all other sets of $n+1$ points from $C\cap\partial B$. By an obvious compactness argument the points $v_1,\dots,v_{n+1}$ exist. We are going to show that the distance above is zero, i.e. $0\in\delta$.

Suppose the contrary. Let $u\in\delta$ satisfy $d(u,0)=d(\delta,0)>0$, where $d(.,.)$ is the distance. Denote by H the half-space such that $0\in H$ and ∂H is orthogonal to the segment $u0$ and passes through its midpoint. Clearly $C\cap\partial B\cap H$ is non-empty, since otherwise one could shift C along $u0$ to a position in $\operatorname{int} B$. Take $v \in C \cap \partial B \cap H$.

If δ is an n-simplex then u belongs to one of its faces, say $v_1 v_2 \cdots v_n$. Now $\operatorname{conv}\{v_1, v_2, \dots, v_n, v\}$ is closer to 0 than δ because it contains the segment uv and angle $vu0$ is less than $\pi/2$. That contradicts the definition of δ.

If δ has a lower dimension, it lies in a hyperplane R^{n-1}. It follows from the Caratheodory Theorem [2, p. 15] that the convex hull of a figure in R^{n-1} can be represented as the union of convex hulls of all n-tuples from the figure. Thus

$$\delta = \bigcup_{i=1}^{n+1} \operatorname{conv}(\{v_1, v_2, \dots, v_{n+1}\} \setminus v_i) \tag{13}$$

and u belongs to one of the sets in this union, say, $u \in \operatorname{conv}\{v_1, v_2, \dots, v_n\}$. Now the set $\operatorname{conv}\{v_1, v_2, \dots, v_n, v\}$ yields a contradiction as above.

Since $\operatorname{diam} \delta \leqq D$, the set δ belongs to $\Delta_n(D)$. Theorem 1.3 (A) implies now that $D \geqq d_n$ and thus 1.2(i) has been proved.

2.2. We prove now 1.2(ii). Let R^k be the affine hull of C. Since $0 \in \delta \subset C \subset R^k$, the set $R^k \cap B$ is a unit ball B_k. Clearly B_k is the smallest ball in R^k containing C (otherwise B could be replaced with a smaller ball). Theorem 1.2(i) (applied to $C \subset B_k$) implies now that $D \geqq d_k$. Together with $D < d_{m-1}$, this yields $d_{m-1} > d_k$; $m - 1 < k$; $m \leqq k$. This establishes 1.2(ii).

2.3. We prove now 1.2(iii). First, exactly as in 2.1, we select points $v_1, v_2, \dots, v_{k+1} \subset C \cap \partial B_k$ such that $\Delta = \operatorname{conv} \bigcup_{i=1}^{k+1} v_i$ contains 0. Since $D \in [d_k, d_{k-1})$ for $k = m$, and by (2), $\Delta \in \Delta_k(D)$. By Theorem 1.3 (B), Δ is a k-simplex and $0 \in \operatorname{int} \Delta$. Thus Δ satisfies (α) and (β) of 1.2(iii). Theorem 1.3 (D) implies that $\lambda_k(D) \leqq l \leqq D$. The example of $C = \sigma_k(D)$ (see 1.3 (C)) shows that both estimates here are unimprovable. Thus Δ satisfies (γ) and Theorem 1.2 has been proved.

§3. Proof of Theorem 1.3

3.1. We use induction by k. We first check that Theorem 1.3 is true for $k = 2$.

Suppose $\Delta_2(D)$ is non-empty for some $D \in (0, 2)$. Clearly, each $\Delta \in \Delta_2(D)$ is a non-degenerate triangle with $0 \in \operatorname{int} \Delta$, since otherwise either Δ itself or one of its sides should be a diameter of the circle B_2 and therefore $\operatorname{diam} \Delta = 2$. Thus 1.3 (B) holds. Let triangle abc belong to $\Delta_2(D)$ with

$$ab \leqq bc \leqq ac \leqq D. \tag{14}$$

Suppose $bc < D$. As a point p moves along ∂B_2 from b to the end c' of the diameter cc', the side pc of the triangle apc grows from $bc < D$ to 2 while the side ap decreases. At a certain position, one has $pc = D$, $ap < ab$, $apc \in \Delta_2(D)$. Since $ap < pc$ and (due to (14)) $ap < ac$, one has $l(apc) = ap < ab = l(abc)$.

Thus a triangle $abc \in \Delta_2(D)$ with the property $l(abc) = \lambda_2(D)$ should satisfy $bc = D$ so that

$$ab \leqq bc = ac = D; \tag{15}$$

$$\lambda_2(D) = ab = D\sqrt{4 - D^2}. \tag{16}$$

Therefore 1.3(C) holds.

Combining (16) with (15), one finds that $D \geqq \sqrt{3}$ if $\Delta_2(D)$ is non-empty. Clearly $\Delta_2(D)$ is non-empty for $D \in [\sqrt{3}, 2)$ because then $s_2 \in \Delta_2(D)$. Since $\lambda_2(\sqrt{3}) = \sqrt{3}$, each element of $\Delta_2(\sqrt{3})$ is isometric to s_2. Thus, 1.3(A) has been established.

Combining (16) with $D \geqq \sqrt{3}$ gives 1.3(D).

3.2. Suppose Theorem 1.3 is true for some $k - 1 \geqq 2$. Then we prove 1.3(B) for k. Let $\Delta_k(D)$ be non-empty for some $D \in (0, d_{k-1})$ and let $\Delta = \operatorname{conv}\{v_1, v_2, \ldots, v_{k+1}\} \in \Delta_k(D)$. Suppose $\dim \Delta < k$. Then Δ lies in a hyperplane R^{k-1}. Similarly to (13),

$$\Delta = \bigcup_{i=1}^{k+1} \operatorname{conv}(\{v_1, v_2, \ldots, v_{k+1}\} \setminus v_i)$$

and 0 belongs to one of the sets in the union, say, $\delta = \operatorname{conv}\{v_1, v_2, \ldots, v_k\}$. Since $v_1, v_2, \ldots, v_k \in \partial B_{k-1} = \partial B_k \cap R^{k-1}$ and $\operatorname{diam} \delta \leqq \operatorname{diam} \Delta \leqq D$, the set δ belongs to $\Delta_{k-1}(D)$. By 1.3(A) (with k replaced by $k-1$), $D \geqq d_{k-1}$, which contradicts the assumption $D \in (0, d_{k-1})$ above. Thus $\dim \Delta = k$.

Suppose $0 \notin \operatorname{int} \Delta$. Then 0 belongs to one of its faces, say $\delta = \operatorname{conv}\{v_1, v_2, \ldots, v_k\}$. That yields a contradiction exactly as above. Thus $0 \in \operatorname{int} \Delta$ and 1.3(B) has been proved.

3.3. We will prove here that for any k-simplex $\Delta = \operatorname{conv}\{v_1, v_2, \ldots, v_k, v\}$ with the vertices on ∂B_k and with $0 \in \operatorname{int} \Delta$ and for any $\varepsilon > 0$ there exists a point $u \in \partial B_k$ such that $\tilde{\Delta} = \operatorname{conv}\{v_1, v_2, \ldots, v_k, u\}$ is a k-simplex with $0 \in \operatorname{int} \tilde{\Delta}$ satisfying:

$$|uv_1 - vv_1| < \varepsilon, \quad uv_i < vv_i, \qquad i = 1, 2, 3, \ldots, k. \tag{17}$$

Since the point $0 \in \operatorname{int} \Delta$, it does not belong to the $(k-2)$-plane containing the $(k-2)$-facet $v_2 v_3 \cdots v_k$. Denote by P the $(k-1)$-plane passing through

$0, v_2, v_3, \dots, v_k$ and by Q the hyperplane tangent to B_k at v. Note that $v \notin P$ because the face $v_2 v_3 \cdots v_k v$ and 0 cannot lie in the same hyperplane P. Thus the segment $0v \perp Q$ does not lie in P and the orthogonal projection $\pi : P \to Q$ is one-to-one. For short, we put $\pi(X) = X'$, $X \subset P$.

Obviously $(\operatorname{conv}\{0, v_2, v_3, \dots, v_k\})' = \operatorname{conv}\{v, v_2', v_3', \dots, v_k'\}$ is a $(k-1)$-simplex in Q. Let vb be its altitude. Then the angle

$$bvv_i' < \pi/2, \qquad i = 2, 3, \dots, k. \tag{18}$$

The minimal geodesic on ∂B_k with non-antipodal ends x, y as well as its length will be denoted by $\breve{xy}$. Pick a point $u \in \partial B_k$ such that the arc $\breve{vu}$ has direction of the segment vb. Since $0 \in \operatorname{int} \Delta$, there are no antipodal pairs among the vertices of Δ. Clearly $\breve{vv_i}$ has the direction of vv_i'. Now (18) implies $\breve{uv_i} < \breve{vv_i}$ and consequently $uv_i < vv_i$ when $\breve{vu}$ is sufficiently short. By continuity, $\tilde{\Delta}$ is a k-simplex with $0 \in \operatorname{int} \tilde{\Delta}$ and $|uv_1 - vv_1| < \varepsilon$ when u is close enough to v.

3.4. We prove now 1.3(C). Let $\Delta \in \Delta_k(D)$ for some $D \in (0, d_{k-1})$ and let $l(\Delta) = \lambda_k(D)$. Let ab be an edge of Δ of length $\lambda_k(D)$. Suppose another of its edges cv satisfies $cv < D$. If ab and cv have a common vertex, say $b = v$, then one can replace v by a nearby point $u \in \partial B_k$ as in 3.3, so that cu is still shorter than D and all other edges coming from the new vertex u become shorter. In particular, $au < av = \lambda_k(D)$, which is impossible since

$$\tilde{\Delta} = \operatorname{conv}(u \cup \operatorname{vert}\Delta \setminus v) \in \Delta_k(D).$$

If ab and cv have no common vertex, we apply first the previous procedure (varying the vertex v) to shorten the edge vb, so that it becomes shorter than D. Then we vary b so as to shorten ab and get the same contradiction as above. Thus $cv = D$ and 1.3(C) has been proved. Note that $\Delta = \sigma_k(D)$ as defined in 1.2, Remark 2.

3.5. We prove now 1.3(D). By 3.4, $\lambda_k(D)$ is the length of the edge ab of the simplex $\Delta = \sigma_k(D)$ there. By Remark (2) in 1.2, $\sigma_k(D)$ is the convex hull of two regular $(k-1)$-simplexes δ' and δ'' with edge length D and with their vertices on ∂B_k. They have a common face δ (a regular $(k-2)$-simplex with edge length D) and the vertices $a \in \delta'$ and $b \in \delta''$ oppose δ. Let c be the center of δ and v be one of its $k-1$ vertices. By reasons of symmetry, the point c and the midpoint e of ab lie on the same diameter of B_k (see Fig. 1).

It is known that

$$cv = D\sqrt{\frac{k-2}{2(k-1)}}; \tag{19}$$

(20) $$ca = D\sqrt{\frac{k}{2(k-1)}}.$$

From the triangle $cv0$ and (19), one gets

(21) $$ce - 0e = \sqrt{1 - D^2\frac{k-2}{2(k-1)}}.$$

(Note that 0 is between c and e since $0 \in \operatorname{int} \sigma_k(D)$.) From the triangle cae and (20), one gets

(22) $$ce^2 + \left(\frac{ab}{2}\right)^2 = D^2\frac{k}{2(k-1)}.$$

From the triangle $0ae$, we have

(23) $$0e^2 + \left(\frac{ab}{2}\right)^2 = 1.$$

Solving the system (21), (22) and (23) with the unknowns ce, $0e$ and ab one gets

(24) $$\lambda_k(D) = ab = D\sqrt{\frac{4k - 2D^2(k-1)}{2-(k-2)(D^2-2)}}.$$

The condition $\lambda_k(D) \leqq D$ implies now

(25) $$D \geqq \sqrt{\frac{2k+2}{k}} = d_k.$$

The relations (24) and (25) constitute 1.3(D).

3.6. We prove now 1.3(A). The inequality (25) implies that $\Delta_k(D)$ is empty for $D < d_k$. It is non-empty for $D \in [d_k, d_{k-1})$ because it contains then s_k. The equalities (24) and (25) imply that

(26) $$\lambda_k(d_k) = d_k.$$

This shows that each $\Delta \in \Delta_k(d_k)$ is isometric to s_k. This completes the proof.

§4. Proof of Lemma 1.3

4.1. Let a triangle $abc \in \Delta_2(D)$ and let

(27) $$ab \leqq bc \leqq ac \leqq D.$$

Suppose $bc < D$. Take a point b' such that the segments bb' and ca are parallel

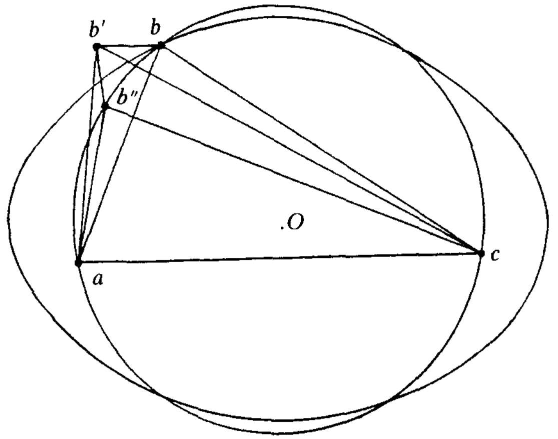

Fig. 3.

and have the same direction (see Fig. 3). Denote by r, r' the inradii of the triangles abc and $ab'c$. Let A be the area of these triangles. Obviously

$$(28) \qquad r = \frac{2A}{ab + bc + ac}; \qquad r' = \frac{2A}{ab' + b'c + ac}.$$

Since $ab \leqq bc$ (see (27)), the point b' lies outside the ellipse passing through b with foci at a and c. Therefore $ab' + b'c > ab + bc$ and, by (28),

$$r' < r.$$

Pick a point $b'' \in \partial B_2 \cap \operatorname{int} abc$ (b'' exists since $b' \notin B_2$). As b' is close enough to b, one has $0 \in \operatorname{int} ab''c$. The angle $ab''c = abc < \pi/2$ so that the angles $ab''b'$ and $cb''b'$ are obtuse and consequently

$$b''c < b'c; \qquad b''a < b'a.$$

Since $b'c < D$ and $b'a < D$ for b' close to b, the triangle $ab''c$ belongs to $\Delta_2(D)$. If r'' is its inradius, then $r'' < r' < r$. Therefore if $r_2(abc) = r = \rho_2(D)$ then the assumption $bc < D$ above is wrong, i.e., $bc = ac = D$ and $abc = \sigma_2(D)$. Elementary calculation shows now that (3) holds.

We leave it to the reader to prove that the lower bound $\omega_2(D)$ is realized by $\sigma_2(D)$ and (4) holds.

4.2. We prove now (5) and (6). Obviously $W_k(\Delta) \geqq 2r_k(\Delta)$, $\Delta \in \Delta_k(D)$. However the equality is impossible here since that would imply easily that Δ has a pair of parallel faces. Let Δ satisfy $W_k(\Delta) = \omega_k(D)$. Then

$$\omega_k(D) = W_k(\Delta) > 2r_k(\Delta) \geqq 2\rho_k(D).$$

Therefore (6) follows from (5).

Take an arbitrary $\Delta \in \Delta_k(D)$. Let $p \in \partial\Delta$ be closest to 0. Clearly $p \in \operatorname{int}\delta$ for a face δ of Δ. If R^{k-1} is the hyperplane containing δ then δ is inscribed in a ball $b_{k-1} = B_k \cap R^{k-1}$ and p is the center of b_{k-1}. Let R be its radius. Denote by δ' the image of δ under a dilatation which turns b_{k-1} into B_{k-1}. Since

$$\operatorname{diam}\delta' = \frac{1}{R}\operatorname{diam}\delta \leqq \frac{D}{R},$$

the simplex $\delta' \in \Delta_{k-1}(D/R)$. By Theorem 1.3(A), $D/R \geqq d_{k-1}$. Therefore

$$p0 = \sqrt{1-R^2} \geqq \sqrt{1-D^2/d_{k-1}^2} = \sqrt{1-D^2(k-1)/2k}.$$

Thus Δ contains the ball of radius $\sqrt{1-D^2(k-1)/2k}$ centered at 0 and (5) has been proved.

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